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FAST TRACK COMMUNICATION

A path integral leading to higher order Lagrangians

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Abstract
We consider a simple modification of standard phase-space path integrals and show that it leads in configuration space to Lagrangians depending also on accelerations.

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Phase-space path integrals usually take the form [1]

$$\int D\vec{q}\,D\vec{p}\,\exp\left(\mathrm{i}\int_0^T \mathrm{d}t[\vec{p}\cdot\dot{\vec{q}} - H(\vec{p},\vec{q})]\right), \tag{1}$$

with border conditions enforced by the type of quantum mechanical amplitude to be evaluated. Such integrals (or their Lagrangian counterparts) suffice for most physical applications, provided the symplectic structure is canonical, $\omega_0 = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$.

In this communication, we would like to consider the following modified path integral:

$$\int D\vec{q}\,D\vec{p}\,\exp\left(\mathrm{i}\int_0^T \mathrm{d}t[\vec{p}\cdot\dot{\vec{q}} - H(\vec{p},\vec{q}) + \theta/2(p_1\dot{p}_2 - p_2\dot{p}_1)]\right), \tag{2}$$

with θ a constant of dimension length-squared. We will subsequently work in two space dimensions and with all indices down, $\vec{q} = (q_1, q_2)$, $\vec{p} = (p_1, p_2)$, for notational simplicity. Standard notation will be used for velocity $v_i = \dot{q}_i \equiv \frac{\mathrm{d}q_i}{\mathrm{d}t}$, acceleration $a_i = \ddot{q}_i \equiv \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2}$ and mass (m). The Planck constant is set to 1 throughout. The above apparently innocuous modification actually amounts to a change in the symplectic structure, $\omega_0 \to \omega = \sum_{i=1}^2 \left(\mathrm{d}p_i \wedge \mathrm{d}q^i + \frac{\theta}{2}\,\mathrm{d}p_i \wedge \mathrm{d}p_j\right)$, and has important consequences discussed below.

The path integral with a modified symplectic structure (2) describes transition amplitudes in noncommutative quantum mechanics, a subject first introduced in [2] and intensively studied in the last decade; see [3–9][2]. It may also present further interest, since the modification in equation (2) is quite simple and not too unnatural (it is a sort of magnetic field, but in momentum space [9]). More precisely (2) describes quantum mechanics with an additional nonvanishing

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[2] We do not attempt to appropriately quote the whole noncommutative mechanics literature in [3, 9].

commutator between coordinates, $[q_1, q_2] = \mathrm{i}\theta$. This theory admits a first principles path integral formulation only in phase space, as detailed in [8]. At the classical level, the extended symplectic structure features an additional nonzero Poisson bracket, $\{q_1, q_2\} = \theta \neq 0$, and the resulting equations of motion do not admit a standard Lagrangian formulation [9].

Nevertheless, one may enforce a (effective) Lagrangian formulation in configuration space by integrating over the momenta in the path integral (2). This process is described here. We first perform the calculation and then discuss the result.

Path integral

We path-integrate over the momenta in (2), to obtain the effective Lagrangian. Starting from the partition function

$$\int Dq_1 Dq_2 Dp_1 Dp_2\, \mathrm{e}^{\mathrm{i}S} \tag{3}$$

with action

$$S = \int_0^T \mathrm{d}t \left[p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{\theta}{2}(p_1\dot{p}_2 - p_2\dot{p}_1) - \frac{p_1^2}{2m} - \frac{p_2^2}{2m} - V(q) \right], \tag{4}$$

we wish to integrate over the momenta p_1, p_2. The potential part $V(q)$ depends only on q_1 and q_2 and plays no role in what follows (the method is valid for any $V(q)$, more precisely for any Hamiltonian with separate quadratic dependence upon momenta). We divide the time interval T in n subintervals $\epsilon = \frac{T}{n}$ ($n \to \infty$ achieves the continuum limit), and choose for simplicity the discrete derivative $v^{(k)} \equiv \dot{x}^{(k)} \equiv \frac{x^{(k+1)} - x^{(k)}}{\epsilon}$; no issues requiring symmetric operations of any kind appear in the following. The relevant part of the discretized action (excluding $V(q)$ for now) becomes

$$\tilde{S} = \sum_{k=0}^{n} \left[\epsilon p_1^{(k)} v_1^{(k)} + \epsilon p_2^{(k)} v_2^{(k)} + \frac{\theta}{2}\left(p_1^{(k)} p_2^{(k+1)} - p_2^{(k)} p_1^{(k+1)}\right) - \epsilon \frac{\left(p_1^{(k)}\right)^2 + \left(p_2^{(k)}\right)^2}{2m} \right]. \tag{5}$$

The clearest way to proceed with the coupled Gaussian integrals is to introduce matrix notation. Define the column vectors

$$V \equiv \epsilon \left(v_1^{(0)}, v_1^{(1)}, \ldots, v_1^{(n)} \ldots, v_2^{(0)}, v_2^{(1)}, \ldots, v_2^{(n)} \ldots\right)^T \tag{6}$$

$$P \equiv \left(p_1^{(0)}, p_1^{(1)}, \ldots, p_1^{(n)} \ldots, p_2^{(0)}, p_2^{(1)}, \ldots, p_2^{(n)} \ldots\right)^T \tag{7}$$

and the matrix

$$J = -a \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot 0 & b & 0 & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot 0 & 0 & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & -b & 0 & \cdot & \cdot 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & -b & \cdot & \cdot 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix},$$

where $a = \frac{\epsilon}{2m}$, $b = \frac{m\theta}{\epsilon}$. Its inverse J^{-1} has the same form as above, but with different entries a', b'; namely $a' = 1/a$ and $b' = -b$ (the off-diagonal part changes sign and the overall factor is reversed). In matrix notation, the discrete action becomes

$$\tilde{S} = P^T V + P^T J P. \tag{8}$$

The coordinate transformation

$$\bar{P} \equiv P + \tfrac{1}{2} J^{-1} V \tag{9}$$

does not change the path integral measure ($D\bar{P} = DP$), and leads to

$$\tilde{S} = \bar{P}^T J \bar{P} - \tfrac{1}{4} V^T J^{-1} V. \tag{10}$$

The first term is now integrated out, and no more dependence upon momenta appears, whereas the second term leads to an exponent of the form (modulo a factor of i)

$$-\frac{1}{4} V^T J^{-1} V = \sum_{k=0}^{n} \left[\epsilon \frac{m}{2} \left(v_1^{(k)}\right)^2 + \epsilon \frac{m}{2} \left(v_2^{(k)}\right)^2 - \frac{\theta m^2}{2} \left(v_1^{(k)} v_2^{(k+1)} - v_2^{(k)} v_1^{(k+1)}\right) \right]. \tag{11}$$

Upon taking the continuum limit $\epsilon \to 0$, our main result follows

$$\int Dq_1 Dq_2 Dp_1 Dp_2 \, \mathrm{e}^{\mathrm{i}S} = N \int Dq_1 Dq_2 \exp\left(\mathrm{i} \int_0^T \mathrm{d}t \, L_{\text{eff}}(q_i, v_i, a_i) \right) \tag{12}$$

with

$$L_{\text{eff}} = \frac{m}{2} \left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\theta m^2}{2} (\dot{q}_1 \ddot{q}_2 - \dot{q}_2 \ddot{q}_1) - V(q_1, q_2) \tag{13}$$

and N a constant not depending on q's. We have reintroduced the potential term, which passed unscathed through equations (4)–(13). The second term in (13) is the correction due to noncommutativity; it depends on velocities *and* accelerations, and has a universal character. Its relative simplicity is striking and somehow unexpected. One is reconforted to find that the Lagrangian (13) was studied by Lukierski *et al* [4] and shown to engender a noncommutative structure. A more detailed discussion follows.

Discussion

As already mentioned, cf [8, 9], the resulting effective Lagrangian could not be a standard one, depending only on coordinates and velocities. Given the complications introduced by noncommutativity, one may have expected *a priori* an involved function, perhaps nonlocal or potential dependent. Remarkably, the effective Lagrangian turned out to be the usual one plus a universal correction depending also on the particle accelerations,

$$\Delta L = -\tfrac{1}{2} \theta m^2 (v_1 a_2 - v_2 a_1), \tag{14}$$

θ denoting the noncommutative scale and m, v_i, a_i denoting the mass, velocity, acceleration, respectively, along the i-axis, of a given particle

The term (14) was previously studied in complete detail in [4], although its appearance can be traced back to earlier developments (cf [5–7]). Lukierski *et al* [4] started from considerations of Galilean invariance in (2+1)-dimensions, and added (14) to a free Lagrangian $\frac{m}{2}\vec{v}^2$, to provide a dynamical realization for a free particle Galilean algebra with one extra central charge. Upon constrained quantization of this higher order action (which thus circumvents the no-go theorem of [9]), noncommutative dynamics was shown to emerge for appropriate choices of canonical variables. Two negative-energy 'internal modes' were proved harmless since they decoupled from the four relevant degrees of freedom. Interactions were subsequently introduced in a constrained way in order to keep the ghosts harmless, and were described by potentials depending on noncommutative coordinates.

We review the derivation in [4] from our perspective, putting more emphasis on the Faddev–Jackiw approach [10], in which the canonical structure is obtained more transparently than in the Dirac formalism. Let us start from the Lagrangian

$$L_{\text{LSZ}} = \frac{m}{2} \left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\theta m^2}{2} (\dot{q}_1 \ddot{q}_2 - \dot{q}_2 \ddot{q}_1) - v(q_1, q_2, \dot{q}_1, \dot{q}_2), \tag{15}$$

with q_1, q_2 commuting, and a more general potential is considered (Lukierski *et al* initially took $v = 0$). We put it into the Hamiltonian form using the Ostrogradski formalism (we use the notation of [4], although they proceed differently at this stage of the Faddeev–Jackiw approach):

$$y_i \equiv \dot{q}_i, \qquad \tilde{p}_i \equiv \frac{\partial L}{\partial \ddot{q}_i} = k\epsilon_{ij} y_j, \qquad k \equiv \frac{\theta m^2}{2}, \tag{16}$$

$$x_i \equiv q_i, \qquad p_i \equiv \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}_i} = m y_i - 2k\epsilon_{ij}\dot{y}_j, \tag{17}$$

$$L - p_i\dot{x}_i + \tilde{p}_i\dot{y}_i = H(x, y, p, \tilde{p}) = \tilde{p}_i \frac{\epsilon_{ij}}{k} p_j - \frac{m\tilde{p}_i^2}{2k^2}. \tag{18}$$

The action $\int \mathrm{d}t L$ is already in the Faddeev–Jackiw form, except for the constraint

$$k y_i + \epsilon_{ij}\tilde{p}_j = 0, \tag{19}$$

which is easily solved:

$$\int \mathrm{d}t L = \int \left(p_i\, \mathrm{d}x_i - \tilde{p}_i \frac{\epsilon_{ij}}{k} d\tilde{p}_j \right) - \int \mathrm{d}t \left(\tilde{p}_i \frac{\epsilon_{ij}}{k} p_j - \frac{m\tilde{p}_i^2}{2k^2} \right). \tag{20}$$

The commutation relations are read out from the (inverse of) symplectic form

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{\tilde{p}_1, \tilde{p}_2\} = k/2. \tag{21}$$

We must now identify the 'true' coordinates X_i of the system. A Noether symmetry analysis immediately unveils that in the $v = 0$ case [4], the Galilean boosts are given by $G_i = p_i t - m x_i + 2\tilde{p}_i \equiv p_i t - K_i$. We have $\dot{G}_i = 0$, $\dot{K}_i = p_i$. Since no time appears in X_i, only the K_i part of G_i matters for the definition of X_i:

$$X_i \equiv \frac{K_i}{m} = x_i - 2\tilde{p}_i/m. \tag{22}$$

Momenta keep the same form, $P_i \equiv p_i$, and their extraneous pair is required to commute with X_i, P_i, leading to $\tilde{P}_i \equiv k p_i/m + \epsilon_{ij}\tilde{p}_j$.

The Hamiltonian reads in the new variables $X_i, P_i, \tilde{P}_i, i = 1, 2$,

$$H = \frac{P_i^2}{2m} - \frac{m\tilde{P}_i^2}{2k^2} + v\left(q_i = X_i + \frac{2\epsilon_{ij}}{m}\left(\frac{k p_j}{m} - \tilde{P}_j\right), \dot{q}_i = \frac{k p_i/m - \tilde{P}_i}{k} \right), \tag{23}$$

whereas the commutators are

$$\{X_1, X_2\} = \frac{2k}{m^2} = \theta, \qquad \{X_i, P_j\} = \delta_{ij}, \qquad \{\tilde{P}_1, \tilde{P}_2\} = k/2. \tag{24}$$

The last two equations define a noncommutative theory. The second term in the Hamiltonian is however negative. To keep it decoupled from the X, P variables, one has to impose that no $\tilde{P}_i$ appear in v. Thus v must depend on the linear combination $q_i - \frac{2k\epsilon_{ij}}{m}\dot{q}_i$ of its variables, which results exactly in the noncommuting coordinates X_i. (LSZ assumed this and proved consistency. We went slightly beyond their work and argued that this condition is also necessary, not only sufficient.)

No obvious reciprocal of the canonical analysis of [4] is known to us at present; actually a classical canonical approach leads to second (not third) order equations of motion [9]. We provided here for the first time a univoque path integral derivation. (The inverse—Lagrangian to Hamiltonian—analysis of [4] indeed suggests (14) as an interesting possibility, as already pointed in [5], but does not single it out. The maximal order of the derivatives appearing in the effective Lagrangian is not fixed *a priori*.) We obtained the additional acceleration-dependent term of [4] up to coefficients, and such a correction turned out to be the only possibility available for noncommutative systems of Heisenberg type and Hamiltonians of the form $H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + V(q_1, q_2)$. Our derivation started *ab initio* with arbitrary potentials $V(q_1, q_2)$, in contrast to the inverse route taken in [4], where the (in the end noncommuting) variables were first carefuly pinned down in the free theory.

The price to be paid for the initial noncommutativity of the coordinates is the appearance of second-order time derivatives in the action, and the ensuing lack of appropriate boundary/initial conditions for the two irrelevant ghost-like additional degrees of freedom. Indeed, the classical equations of motion engendered by (13) are of third order in time derivatives,

$$\epsilon_{ij}\theta m^2 \frac{\mathrm{d}^3 q_j}{\mathrm{d}t^3} + m\ddot{q}_i + \partial_{q_i} V = 0. \tag{25}$$

No fourth-order time derivatives arise for q_1, q_2, and this leads to two constraints in the Hamiltonian formulation. Six constants are required—two more in comparison with the commutative case; only four are available (for instance, the initial and final values of q_1 and p_2). This apparent indeterminacy is a consequence of the initial noncommutativity of q_1 and q_2, but poses no serious problem. The missing two constants are actually needed to specify the motion of the two 'internal' modes, modes which must be eliminated for consistency, cf [4] (see also [7]).

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